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The evaluation of the conjugate function of a periodic spline on a uniform mesh *

Martin H. GUTKNECHT
Seminar für angewandte Mathematik, Eidgenössische Technische Hochschule Zürich, CH-8092 Zürich, Switzerland

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Abstract: We present an $O(N \log N)$ algorithm for the simultaneous evaluation (at all N points of a uniform mesh) of a function conjugate to a periodic spline. The algorithm is based on the attenuation factor theory and makes use of the FFT. It can be applied to other function classes with known attenuation factors also. For the approximation by functions conjugate to splines of odd degree we state several optimality criterions and error bounds.

Keywords: Conjugate function, discrete Hilbert transform, attenuation factors, periodic splines.

1. Introduction

Let L_2 denote the Hilbert space of square integrable 2π-periodic real functions. For $X \in L_2$ with complex Fourier coefficients γ_k the *conjugate periodic function* (or *Hilbert transform* with respect to the circle) $KX \in L_2$ is defined by having the Fourier coefficients

$$\delta_k = -\mathrm{i}\,\mathrm{sign}(k)\gamma_k, \quad \forall k \in \mathbb{Z}, \tag{1.1}$$

cf., e.g., [22]. (We always use the complex Fourier coefficients although numerical computations can be done more efficiently with the real ones.) K is called the *conjugation operator*. The standard numerical method for computing KX is based on the interpolation of X at $N = 2n$ equidistant points by a trigonometric polynomial T of degree n. Its coefficients can be obtained by a fast Fourier transform (FFT), another application of which yields simultaneously all the values of KT at the same meshpoints [7,17,20,21]. Unfortunately, the approximation of KX by KT may be very poor if the Fourier series of X does not converge fast enough.

In this paper we point out that for many other classes of approximants ϕ the conjugate periodic function $K\phi$ and its derivatives can be evaluated on the given uniform mesh or a finer one nearly as efficiently, provided that some constants κ_k (depending on the class and the mesh, but not on the data) have been computed in advance. Our approach is based on the theory of *attenuation factors* (see [13] and the references given there, including [4,12,14,28]). In particular, it applies to (possibly deficient) periodic spline interpolants and to smoothing by spline functions. For both cases we state explicit formulas for the factors κ_k (Sections 5 and 6). Moreover, for

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these classes we establish in Section 7 the optimality of $K\phi$ in the sense of the theories of Sard–Schoenberg [31,33], Golomb–Weinberger [16], and Schoenberg–Reinsch [29,30,31,34], respectively. (Optimal properties of certain other approximations to the conjugation operator have been discussed by Brass [6] and Knauff and Kress [23].)

Error bounds for the conjugate function of spline interpolants and its derivatives are given in Section 9. In particular, we verify that most of Golomb's bounds in [14] remain valid.

Further results given include a necessary and sufficient condition for $K\phi$ to have the same attenuation factors as ϕ (Section 4) and two methods to deal with functions X having step singularities (Section 8).

The conjugation process presented here, with spline interpolants or smoothing splines as approximants, has been applied with great success in the numerical solution of Theodorsen's integral equation of conformal mapping, see [18] for numerical experiments. Other possible applications include: related methods of conformal mapping [19], the numerical evaluation of Schwarz's integral, and the solution of Riemann–Hilbert problems on the disk [19].

2. Prerequisites: attenuation factors

Let Π_N^{R} be the real N-dimensional vector space of N-periodic real sequences $x = (x_k)_{k=-\infty}^{\infty}$, and Π_N^{S} the real vector space of N-periodic conjugate even sequences $c = (c_k)_{k=-\infty}^{\infty}$ (with $c_k = \bar{c}_{-k} \in \mathbb{C}$). The *discrete Fourier transform (DFT)* [8,21]

$$\mathscr{F}_N\colon\ x \mapsto c, \quad \text{where} \quad c_k := \frac{1}{N}\sum_{j=1}^{N} x_j\, \mathrm{e}^{-2\pi \mathrm{i} jk/N}, \quad \forall k \in \mathbb{Z}, \tag{2.1}$$

is an isomorphism of Π_N^{R} onto Π_N^{S}. The FFT allows one to execute $\mathscr{F}_N$ and $\mathscr{F}_N^{-1}$ by only $\mathrm{O}(N \log N)$ multiplications.

By Φ denote the real Wiener algebra of continuous real valued 2π-periodic functions having an absolutely convergent Fourier series. The nondiscrete Fourier analysis operator $\mathscr{F}$ maps Φ one-to-one onto the subspace of conjugate even sequences $\gamma = (\gamma_k)_{k=-\infty}^{\infty} \in l_1$ of Fourier coefficients of $\phi \in \Phi$,

$$\gamma_k := \frac{1}{2\pi}\int_0^{2\pi} \phi(s)\, \mathrm{e}^{-\mathrm{i}ks}\, \mathrm{d}s, \quad \forall k \in \mathbb{Z}.$$

On Π_N^{R} and Φ the shift operator E is defined by

$$(Ex)_k := x_{k+1}, \qquad \forall k \in \mathbb{Z}, \quad \forall x \in \Pi_N^{\mathrm{R}}, \tag{2.2a}$$

$$(E\phi)(s) := \phi(s + \pi/N), \qquad \forall s \in \mathbb{R}, \quad \forall \phi \in \Phi, \tag{2.2b}$$

respectively.

Let $P\colon\ \Pi_N^{\mathrm{R}} \to \Phi$ always be a linear operator with the following properties:

(T) P is translation invariant: $PEx = EPx$, $\forall x \in \Pi_N^{\mathrm{R}}$;

(S) P preserves symmetry about 0, i.e. Px is an even (odd) function if x is an even (odd) sequence;

(U) $P(\ldots, 1, 1, 1, \ldots)^{\mathrm{T}} \equiv 1$.

Then, owing to property (T), the DFT $c := \mathscr{F}_N x$ of x and the Fourier coefficients $\gamma := \mathscr{F}\phi$ of $\phi := Px$ are related by [13, Theorem 3.1]

$$\gamma_k = \tau_k c_k, \quad \forall k \in \mathbb{Z}, \tag{2.3}$$

where the *attenuation factors* τ_k are independent of x:

$$\tau := (\tau_k) := N\mathscr{F}Pe \in l_1 \tag{2.4}$$

with $e := (e_k)$, $e_k := 1$ if $k \equiv 0$ (mod N), $e_k := 0$ otherwise. According to Remark 2 in [13, p. 382], which remains true if P is not an interpolation operator, assumption (U) implies

$$\tau_0 = 1, \quad \tau_{jN} = 0, \qquad \forall j \in \mathbb{Z} \setminus \{0\}. \tag{2.5}$$

From (S) one concludes (cf. [13, Remark 3])

$$\tau_k = \tau_{-k} \in \mathbb{R}, \quad \forall k \in \mathbb{Z}. \tag{2.6}$$

Relation (2.3) is even characteristic for a linear translation invariant operator P: If P: $\Pi_N^{\mathrm{R}} \to \Phi$ is an operator such that (2.3) holds for any $\gamma := \mathscr{F}Px$ and $c := \mathscr{F}_N x$, generated by some $x \in \Pi_N^{\mathrm{R}}$, and if the coefficients τ_k are independent of x, then P is necessarily linear and translation invariant [13, Theorem 3.2].

Let us denote by $\theta := (\theta_k)_{k=-\infty}^{\infty}$ the mesh of equidistant points $\theta_k := 2\pi k/N$. In the following $x \in \Pi_N^{\mathrm{R}}$ will often be a sequence of values $x_k := X(\theta_k)$ of some 2π-periodic function X evaluated on θ. P is called *interpolation operator* iff $(Px)(\theta_k) = x_k$, $\forall k \in \mathbb{Z}$, or, briefly, $(Px)(\theta) = x$. For an interpolation operator, the well-known formula (e.g., [13, p. 381]),

$$c_k = \sum_{j=-\infty}^{\infty} \gamma_{k+jN}, \quad \forall k \in \mathbb{Z}, \tag{2.7}$$

holds, which explains the aliasing effect. Relation (2.3) then implies

$$\sum_{j=-\infty}^{\infty} \tau_{k+jN} = 1, \quad \forall k \in \mathbb{Z}. \tag{2.8}$$

For example, if ϕ is the broken line interpolant, then $\tau_k = (\sin \theta_k/\theta_k)^2$. Many more examples are given in [13]; some of them are cited below.

3. The evaluation of the conjugate function on a mesh of equidistant points

For $m \geqslant 0$ let C^m denote the space of m times continuously differentiable 2π-periodic real functions. Set $\mathrm{D}^m := \mathrm{d}^m/\mathrm{d}t^m$ and $\Phi^m := \{X \in C^m : \mathrm{D}^m X \in \Phi\}$. Moreover, let $n := \lfloor N/2 \rfloor$ (i.e., $N = 2n$ if N is even and $N = 2n + 1$ if N is odd), and, for some fixed positive integer ν, let $n' := \lfloor \nu N/2 \rfloor$. Also, let P (defined in Section 2) now satisfy $P(\Pi_N^{\mathrm{R}}) \subset \Phi^m$ for some $m \geqslant 0$. By making use of the attenuation factor theory we readily obtain a simple but fast algorithm for the simultaneous evaluation of the function $\mathrm{D}^m KPx$ at all points of the uniform mesh

$$\theta' := (\theta_k')_{k=-\infty}^{\infty}, \quad \theta_k' := k\pi/\nu N, \qquad \forall k \in \mathbb{Z},$$

which is either identical with (if $\nu = 1$) or finer than the mesh θ.

As in Section 2 we let $\phi := Px$, $c := \mathscr{F}_N x$, and $\gamma := \mathscr{F}\phi$. According to (1.1) and (2.3) $\mathrm{D}^m K\phi$ has the Fourier coefficients

$$\delta_k := \delta_k^{(m)} := \mathrm{i}^{m-1} k^m \operatorname{sign}(k) \gamma_k = \mathrm{i}^{m-1} k^m \operatorname{sign}(k) \tau_k c_k, \quad \forall k \in \mathbb{Z}. \tag{3.1}$$

On the other hand, it is well known that the trigonometric polynomial of degree n' interpolating $\mathrm{D}^m K\phi$ on θ' (and normalized by a real n'th coefficient if νN is even) is

$$\sum_{k=-n'}^{n'}{}' \, d_k \mathrm{e}^{\mathrm{i}kt}, \tag{3.2}$$

where the prime at the summation symbol indicates that the terms with indices $\pm n'$ have weight $\frac{1}{2}$ if νN is even, and where due to (2.7)

$$d_k := d_k^{(m)} := \sum_{j=-\infty}^{\infty} \delta_{k+j\nu N}, \quad \forall k \in \mathbb{Z}. \tag{3.3}$$

Note that $d := (d_k) \in \Pi_{\nu N}^{\mathrm{S}}$, and that the sequence

$$y := (\mathrm{D}^m K\phi)(\theta') := \left((\mathrm{D}^m K\phi)(\theta_k')\right)_{k=-\infty}^{\infty} \in \Pi_{\nu N}^{\mathrm{R}} \tag{3.4}$$

is obtained by evaluating (3.2) on θ', which can be done with the FFT:

$$y = \mathscr{F}_{\nu N}^{-1} d. \tag{3.5}$$

Theoretically, the problem of evaluating $\mathrm{D}^m K\phi$ on θ' is solved by (3.1), (3.3), and (3.5). However, we can replace the infinite sum in (3.3) by a relation that involves an infinite sum of terms independent of ϕ. Moreover, for spline interpolation operators P we will be able to compute this sum analytically.

According to (3.1), (3.3) and the periodicity of c

$$d_k = \kappa_k c_k, \quad \forall k \in \mathbb{Z}, \tag{3.6}$$

with

$$\kappa_k := \kappa_k^{(m)} := \mathrm{i}^{m-1} \sum_{j=-\infty}^{\infty} (k + j\nu N)^m \operatorname{sign}(k + j\nu N) \tau_{k+j\nu N}, \quad \forall k \in \mathbb{Z}. \tag{3.7}$$

Here the sum converges absolutely since $P(\Pi_N^{\mathrm{R}}) \subset \Phi^m$. In view of $c \in \Pi_N^{\mathrm{S}} \subset \Pi_{\nu N}^{\mathrm{S}}$ and $d \in \Pi_{\nu N}^{\mathrm{S}}$, we have $\kappa \in \Pi_{\nu N}^{\mathrm{S}}$. Obviously $\mathrm{i}^{m-1}\kappa \in \Pi_{\nu N}^{\mathrm{R}}$ too, so that

$$\kappa_{-k} = \bar{\kappa}_k = (-1)^{m-1} \kappa_k, \quad \forall k \in \mathbb{Z}. \tag{3.8}$$

By (2.5) and (3.7) we also get

$$\kappa_{jN} = 0, \quad \forall j \in \mathbb{Z}. \tag{3.9}$$

Finally, due to (2.6) and (3.7)

$$\kappa_{j\nu n} = 0, \qquad \forall j \in \mathbb{Z}, \quad \text{if } m \text{ and } N \text{ are even.} \tag{3.10}$$

We summarize our results in

Theorem 3.1. *Let $m \geq 0$ and assume that P: $\Pi_N^{\mathrm{R}} \to \Phi^m$ is a linear operator satisfying conditions (S), (T), and (U); call the associated attenuation factors τ_k.*

Then, for any $x \in \Pi_N^{\mathrm{R}}$ the sequence y [specified in (3.4)] of values which the mth derivative of the conjugate approximation function KPx attains on the uniform mesh θ' can be computed according to $c := \mathscr{F}_N x$, and (3.5)–(3.7). Only two fast Fourier transforms (of length N and νN, respectively) are needed, provided the factors κ_k, which depend only on P, N, ν, and m, have been precomputed. Since $\kappa \in \Pi_{\nu N}^{\mathrm{S}}$, evaluating (3.7) is only necessary for $k = 1, \ldots, n'$.

Remark. The second algorithm presented in [17, Section 6] for the evaluation of the conjugate trigonometric interpolant on even or odd mesh points could be extended to $\mathrm{D}^m KPx$ if and only if $\kappa_k = \kappa_{n-k}$, $k = 1, \ldots, n-1$. (N even and $\nu = 1$ are required anyway.) However, this case is of little practical interest.

Gautschi [13] extended the attenuation factor theory also to the case where values $x_k^{(j)} := \mathrm{D}^j X(\theta_k)$, $j = 1, \ldots, q-1$, of the first $q-1$ derivatives of $X \in \mathrm{C}^{q-1}$ are given in addition to the function values $x_k^{(0)} := x_k := X(\theta_k)$. We consider an operator P: $(\Pi_N^{\mathrm{R}})^q \to \Phi^m$ $(m \geqslant 0)$ associating with the given data $x = (x^{(j)})_{j=0}^{q-1}$ a function $\phi = Px \in \Phi^m$. (Here $x^{(j)} = (x_k^{(j)})_{k=-\infty}^{\infty} \in \Pi_N^{\mathrm{R}}$, $j = 0, \ldots, q-1$.) If P is linear and translation invariant [defined by a natural extension of condition (T)], then there exist attenuation factors $\tau_{k,j}$ such that the Fourier coefficients $\gamma := (\gamma_k) := \mathscr{F}\phi$ of ϕ satisfy

$$\gamma_k = \sum_{j=0}^{q-1} \tau_{k,j} c_k^{(j)}, \quad \forall k \in \mathbb{Z}, \tag{3.11}$$

where $c^{(j)} := (c_k^{(j)}) := \mathscr{F}_N x^{(j)}$. To evaluate $\mathrm{D}^m K\phi$ on the mesh θ' we can again make use of (1.1) and (3.3)–(3.5). By separating the infinite sum that is independent of the data we now get in analogy to (3.5)–(3.7)

$$y := K\phi(\theta') = \mathscr{F}_{\nu N}^{-1} d, \tag{3.12}$$

$$d_k := \sum_{j=0}^{q-1} \kappa_{k,j}^{(m)} c_k^{(j)}, \quad \forall k \in \mathbb{Z}, \tag{3.13}$$

$$\kappa_{k,j}^{(m)} := \mathrm{i}^{m-1} \sum_{l=-\infty}^{\infty} (k + l\nu N)^m \operatorname{sign}(k + l\nu N) \tau_{k+l\nu N, j},$$
$$\forall k \in \mathbb{Z}, \quad j = 0, \ldots, q-1. \tag{3.14}$$

4. Attenuation factors of the conjugate function

With respect to the basis $\{\mathrm{e}^{iks}\}_{k=-\infty}^{\infty}$ of L_2 the operators K and E are both diagonal, and hence they commute. (The same is true for D, but D does not map L_2 or Φ into itself.) Consequently, KP satisfies (T) also, and in fact, according to (3.1) KP has the attenuation factors $-\mathrm{i}\operatorname{sign}(k)\tau_k$, which relate c_k and δ_k.

However, are there also attenuation factors τ_k' which allow us to reconstruct δ_k from d_k? This would mean that there is a translation invariant operator P' such that for any $x \in \Pi_N^{\mathrm{R}}$

$$KPx = P'y, \quad \text{where} \quad y := (KPx)(\theta'). \tag{4.1}$$

From the diagram

$$\begin{array}{ccc} c = \{c_k\} \in \Pi_N^S & \overset{\cdot \tau_k}{\mapsto} & \gamma = \{\gamma_k\} \in l_1 \\ \downarrow \cdot \kappa_k & & \downarrow \cdot -i\operatorname{sign}(k) \\ d = \{d_k\} \in \Pi_{\nu N}^S & \overset{\cdot \tau_k'}{\mapsto} & \delta = \{\delta_k\} \in l_1 \end{array} \tag{4.2}$$

we see immediately that τ_k' must satisfy $\tau_k'\kappa_k = -\mathrm{i}\ \mathrm{sign}(k)\tau_k$. Using (2.5) and (3.9) we thus obtain parts (i) and (ii) of the following theorem.

Theorem 4.1. (i) *The Fourier coefficients δ_k of KPx are (for arbitrary data $x \in \Pi_N^{\mathrm{R}}$) reconstructible from the DFT coefficients d_k if and only if*

$$\kappa_k = 0, \quad k \neq 0 \ \Rightarrow\ \tau_k = 0, \tag{4.3}$$

or, equivalently, if and only if

$$\kappa_k = 0, \quad k \not\equiv 0 \pmod N \ \Rightarrow\ \tau_k = 0. \tag{4.4}$$

(ii) *If* (4.3) *holds, then*

$$\delta_k = \tau_k' d_k, \quad \forall k \in \mathbb{Z}, \tag{4.5}$$

where

$$\tau_k' := -\mathrm{i}\ \mathrm{sign}(k)\frac{\tau_k}{\kappa_k} = \frac{\mathrm{sign}(k)\tau_k}{\sum_{j=-\infty}^{\infty} \mathrm{sign}(k+j\nu N)\tau_{k+j\nu N}} \quad \text{if} \quad \kappa_k \neq 0, \tag{4.6}$$

while τ_k' can be chosen arbitrarily if $\kappa_k = 0$.

(iii) *If* (4.3) *and* (4.6) *hold and if we choose $\tau_0' := 1$, $\tau_{j\nu N}' := 0$ if $j \neq 0$, and $\tau_k' = \tau_{-k}'$ as an arbitrary real number in all other cases where $\kappa_k = 0$, then the sequence $\{\tau_k'\}$ is the sequence of attenuation factors of a linear operator P': $\Pi_{\nu N}^{\mathrm{R}} \to \Phi$ satisfying* (S), (T), (U), *and* (4.1).

Proof. There remains to prove (iii). It is easy to verify that (2.5) implies (U) and (2.6) implies (S). Moreover, it follows readily from (4.6) that $\tau_k' = \tau_{-k}'$.

Of particular interest would be the case where $P' = P$. It is characterized in

Theorem 4.2. *Assume* (4.3) *and $\nu = 1$, and choose $\tau_k' := \tau_k$ if $\kappa_k = 0$. Then $\tau_k' = \tau_k$, $\forall k \in \mathbb{Z}$ if and only if*

$$\tau_k \neq 0 \quad \textit{with} \quad k = k_0 + l_0 N, \quad 0 < k_0 < N, \quad l_0 \geqslant 0, \tag{4.7}$$

implies

$$\tau_{lN-k_0} = \tau_{k_0 - lN} = 0, \quad l = 1, 2, \ldots, \tag{4.8}$$

and

$$1 = \sum_{j=0}^{\infty} \tau_{k_0+jN} = \sum_{j=-\infty}^{\infty} \tau_{k_0+jN}. \tag{4.9}$$

Proof. Assume $\tau_k = \tau_k'$. According to (4.3) $\tau_k \neq 0$ implies $\kappa_k \neq 0$ if $k \neq 0$, and by (4.6) one then has

$$\mathrm{sign}(k) = \sum_{j=-\infty}^{\infty} \mathrm{sign}(k+jN)\tau_{k+jN}. \tag{4.10}$$

Here, in contrast to the left-hand side, the right-hand side does not change if we replace $k \neq 0$ by $k - lN$ with $\mathrm{sign}(k - lN) = -\mathrm{sign}(k)$. So, (4.10) only holds either for k or for $k - lN$. Hence, (4.7) implies (4.8), and (4.10) simply becomes (4.9). On the other hand, if for some $k \neq 0$ satisfying (4.7) the relations (4.8) and (4.9) hold, then (4.10) is satisfied for this k (with $\tau_k \neq 0$), and thus $\tau_k = \tau_k'$ by (4.6). Moreover, if $\tau_k = 0$, then $\tau_k' = 0$ due to (4.6) (if $\kappa_k \neq 0$) or by definition (if $\kappa_k = 0$). Thus, $\tau_k' = \tau_k$, $\forall k \neq 0$. Finally, $\tau_0' = \tau_0 = 1$, cf. (2.5).

Note that $P' = P$ requires $\tau_{n+jN} = 0$, $\forall j \in \mathbb{Z}$, if N is even. In fact, (4.7) with $k_0 = n$ contradicts (4.9). This leads to the following corollary.

Corollary 4.3. *If N is even, then there is no interpolation operator P satisfying* (S), (T), (U), *and* $P' = P$.

Proof. If P is interpolary, (2.8) holds. However, $P = P'$ would imply $\tau_{n+jN} = 0$, $\forall j \in \mathbb{Z}$, which yields a contradiction.

For odd N the operator P associated to trigonometric interpolation satisfies (S), (T), (U), and $P' = P$, cf. Example 5.1 below.

5. Examples of interpolation operators P

Example 5.1. Interpolation by a trigonometric polynomial T of degree $n = \lfloor N/2 \rfloor$ (and normalized by a vanishing $\sin nt$ term if N is even). The attenuation factors are $\tau_k = 1$ if $|k| < N/2$, $\tau_k = 0$ if $|k| > N/2$, and, if N is even, $\tau_n = \frac{1}{2}$. In the case $\nu = 1$ we get for N and m even

$$\kappa_k^{(m)} = \begin{cases} 0 & \text{if } k \equiv 0 \pmod{n}, \\ \mathrm{i}^{m-1} \operatorname{sign}(k_0) k_0^m & \text{if } k \equiv k_0 \pmod{N}, \quad 1 < |k_0| < n, \end{cases}$$

while for N odd, or N even and m odd,

$$\kappa_k^{(m)} = \begin{cases} 0 & \text{if } k \equiv 0 \pmod{N}, \\ \mathrm{i}^{m-1} \operatorname{sign}(k_0) k_0^m & \text{if } k \equiv k_0 \pmod{N}, \quad 1 < |k_0| \leqslant n. \end{cases}$$

Finally, assuming $m = 0$ and $\nu = 1$ we get for N even: $\tau_k' = \tau_k$ if $|k| \neq n$, but $\tau_{\pm n}' = 0$, and hence $P' \neq P$. On the other hand, $P' \neq P$ if N is odd.

Example 5.2. Interpolation by a periodic spline function $S_{N,2r}$ of even order $2r$, i.e. odd degree $2r - 1$ $(r \geqslant 1)$. For any $x \in \Pi_N^{\mathrm{R}}$ the interpolating spline function $S := S_{N,2r} = Px$ is defined by

$$S \in C^{2r-2}, \quad S(\theta) = x, \quad \mathrm{D}^{2r} S(t) = 0, \qquad \forall t \notin \theta. \tag{5.1}$$

The associated attenuation factors are $\tau_0 = 1$, $\tau_{jN} = 0$ $(\forall j \neq 0)$, and

$$\tau_k = \left[\sum_{j=-\infty}^{\infty} (1 + jN/k)^{-2r} \right]^{-1}, \quad \forall k \not\equiv 0 \pmod{N}. \tag{5.2}$$

(See, e.g., [13], Example 5.1.) In terms of the functions

$$\sigma_m(z) := \sum_{j=-\infty}^{\infty} \left(\frac{z}{j+z} \right)^{m+1}, \qquad \forall z \notin \mathbb{Z}, \quad m = 0, 1, 2, \ldots$$

introduced by Ehlich [12] and further investigated by Gautschi [13] these factors can be written

$$\tau_k = 1/\sigma_{2r-1}(z_k), \quad \text{where} \quad z_k := k/N, \quad \forall k \not\equiv 0 \pmod N). \tag{5.3}$$

Gautschi [13, Proposition 2.2] has shown that explicitly

$$\sigma_0(z) = \pi z \cot \pi z,$$

$$\sigma_m(z) = \left(\frac{\pi z}{\sin \pi z}\right)^{m+1} q_{m-1}(\cos \pi z), \quad m = 1, 2, \ldots, \tag{5.4a}$$

where q_{m-1} is a polynomial of degree $m-1$ that is recursively defined by

$$q_0(t) := 1, \qquad q_l(t) := tq_{l-1}(t) + \frac{1-t^2}{l+1} q'_{l-1}(t), \quad l = 1, 2, \ldots \tag{5.4b}$$

q_m is even (odd) if m is even (odd).

In view of (3.7) and

$$\tau_{k+j\nu N} = \left(\frac{\sin \pi z_k}{\pi z_{k+j\nu N}}\right)^{2r} \frac{1}{q_{2r-2}(\cos \pi z_k)} = \left(\frac{z_k}{z_{k+j\nu N}}\right)^{2r} \tau_k = \left(\frac{k}{j\nu N + k}\right)^{2r} \tau_k, \tag{5.5}$$

we get for $1 \leqslant k < \nu N$, $0 \leqslant l \leqslant 2r-2$,

$$\kappa_k^{(l)} = \mathrm{i}^{l-1} k^l \tau_k \left[1 + \psi_{2r-l-1}(z'_k) - (-1)^l \psi_{2r-l-1}(-z'_k)\right], \tag{5.6}$$

where

$$z'_k := k/\nu N, \quad \forall k \in \mathbb{Z},$$

$$\psi_m(z) := \sum_{j=1}^{\infty} \left(\frac{z}{j+z}\right)^{m+1}, \quad \forall z \in \mathbb{C} \setminus \{-1, -2, \ldots\}, \qquad m = 1, 2, \ldots . \tag{5.7}$$

Note that σ_m and ψ_m are related by

$$\sigma_m(z) = 1 + \psi_m(z) + \psi_m(-z), \tag{5.8}$$

so that there is no need to evaluate $\psi_m(-z)$ directly. It is also easy to verify that Proposition 2.1 in [13] and its proof hold for the functions ψ_m too,

$$\psi_{m+1}(z) = \psi_m(z) - \frac{z}{m+1} \psi'_m(z), \qquad \forall z \in \mathbb{C} \setminus \{-1, -2, \ldots\}, \quad m = 1, 2, \ldots .$$

However, these functions are not elementary. In fact, ψ_m is closely connected with the *polygamma function* $\psi^{(m)}$ [1, Section 6.4], which is the mth derivative of the psi function [1, Section 6.3; 25, Vol. 1, p. 12] defined by

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \forall z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},$$

where Γ denotes the gamma function. For $z \neq 0, -1, -2, \ldots$

$$\psi_m(z) = z^{m+1} \sum_{j=0}^{\infty} (z+j)^{-m-1} - 1 = \frac{(-1)^{m+1}}{m!} z^{m+1} \psi^{(m)}(z) - 1. \tag{5.9}$$

According to the recurrence formula for $\psi^{(m)}$ [1],

$$\psi^{(m)}(z+1) = \psi^{(m)}(z) + (-1)^m m! z^{-m-1}, \tag{5.10}$$

we finally get

$$\psi_m(z) = \frac{(-1)^{m+1}}{m!} z^{m+1} \psi^{(m)}(z+1), \quad \forall z \in \mathbb{C} \setminus \{-1, -2, \dots\}. \tag{5.11}$$

This formula is more convenient than (5.9) since we have to apply it for $z = z_k' \in (0, \frac{1}{2})$, $k = 1, \dots, n\nu - 1$, and $\psi^{(m)}$ has a pole of order $m+1$ at $z = 0$. Luke [25, Vol. 1, p. 28; Vol. 2, p. 301] describes how to obtain expansions for $\psi^{(m)}$ for the interval [3.0, 4.0] in terms of Chebyshev polynomials and lists the corresponding coefficients for $m = 0, \dots, 6$. We may use these expansions together with (5.6) and (5.8)–(5.10) to compute κ_k, $k = 1, \dots, \lfloor \nu N/2 \rfloor$, efficiently.

Example 5.3. Interpolation by a periodic spline $S := S_{N,2r+1}$ of odd order $2r+1$, i.e. even degree $2r$ ($r \geqslant 0$, but $\tau \in l_1$ only if $r \geqslant 1$). In order to keep property (S) of P, one has to choose the knots midway between the mesh points θ_k,

$$S \in C^{2r-1}, \qquad S(\theta) = x, \qquad \mathrm{D}^{2r+1} S(t) = 0 \quad \text{if } t - \tfrac{1}{2}\theta_1 \notin \theta. \tag{5.12}$$

By a derivation paralleling the one for odd degree splines in [13] and in the previous example we obtain instead of (5.2) to (5.6):

$$\tau_k = \left[\sum_{j=-\infty}^{\infty} (-1)^j (1 + jN/k)^{-2r-1} \right]^{-1} = \tilde{\sigma}_{2r}^{-1}(z_k), \quad \forall k \not\equiv 0 \pmod{N}, \tag{5.13}$$

$$\tilde{\sigma}_m(z) := \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{z}{j+z} \right)^{m+1} = \left(\frac{\pi z}{\sin \pi z} \right)^{m+1} \tilde{q}_m(\cos \pi z),$$

$$\forall z \notin \mathbb{Z}, \quad m = 0, 1, \dots, \tag{5.14a}$$

$$\tilde{q}_0(t) := 1, \qquad \tilde{q}_l(t) := t \tilde{q}_{l-1}(t) + \frac{1-t^2}{l} \tilde{q}_{l-1}'(t), \quad l = 1, 2, \dots, \tag{5.14b}$$

$$\tau_{k+j\nu N} = (-1)^{j\nu} \left(\frac{k}{j\nu N + k} \right)^{2r+1} \tau_k, \quad \forall k \not\equiv 0 \pmod{N}, \tag{5.15}$$

$$\begin{aligned} \kappa_k^{(l)} = \mathrm{i}^{l-1} k^l \tau_k \big[& 1 + (-1)^{\nu} \psi_{2r-l}(z_k') + \{1 - (-1)^{\nu}\} \psi_{2r-l}(\tfrac{1}{2} z_k') \\ & - (-1)^{l+\nu} \psi_{2r-l}(-z_k') - \{(-1)^l - (-1)^{l+\nu}\} \psi_{2r-l}(-\tfrac{1}{2} z_k') \big], \\ & 1 \leqslant k < \nu N, \quad 0 \leqslant l \leqslant 2r - 1, \quad r \geqslant 1. \end{aligned} \tag{5.16}$$

According to (5.2) $\tau_k > 0$, $\forall k \not\equiv 0 \pmod{N}$ in the odd degree case. Here, by writing (5.13) as an alternating series with monotonically decreasing terms, one concludes that $\tau_k > 0$ for $|k| < N$. However, (5.15) shows that $\tau_k < 0$ if $\lfloor |k|/N \rfloor$ is odd.

Example 5.4. Piecewise cubic interpolation. For $x \in \Pi_N^{\mathrm{R}}$ we define $\phi = Px \in \Phi$ by $\phi(t) = p_j(t)$ for all $t \in [\theta_j, \theta_{j+1}]$, where p_j is the cubic polynomial satisfying $p_j(\theta_k) = x_k$, $k = j-1, j, j+1, j+2$. The corresponding attenuation factors are [13, Example 5.2]

$$\tau_k = \left(\frac{\sin \pi z_k}{\pi z_k} \right)^4 \left[1 + \tfrac{2}{3} (\pi z_k)^2 \right], \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{5.17}$$

A short calculation yields for $1 \leqslant k < \nu N$

$$\kappa_k^{(0)} = -\mathrm{i}\left(\frac{\sin \pi z_k}{\pi z_k}\right)^4 \left\{1 + \psi_3(z_k') - \psi_3(-z_k') + \tfrac{2}{3}(\pi z_k)^2\left[1 + \psi_1(z_k') - \psi_1(-z_k')\right]\right\}. \tag{5.18}$$

Example 5.5. Interpolation by a periodic spline $S := S_{N,2r,q}$ of order $2r$ and deficiency q $(1 \leqslant q \leqslant r)$ is defined by

$$\begin{aligned} &S \in C^{2r-1-q}, \qquad \mathrm{D}^j S(\theta) = x^{(j)}, \quad j = 0, \dots, q-1, \\ &\mathrm{D}^{2r} S(t) = 0, \quad \forall t \notin \theta. \end{aligned} \tag{5.19}$$

From its attenuation factors given by Gautschi [13, Example 5.6] one can see that it is again possible to sum up the infinite series in (3.14) analytically and to express it in terms of values $\psi_l(z_k')$ and $\psi_l(-z_k')$ of the functions ψ_l defined by (5.7).

6. Examples of noninterpolary operators based on smoothing

As is well known, a periodic function $\phi \in L_2[0, 2\pi]$ can be smoothed by convolution with a suitable even *weight* or *window function* $W \in \Phi$,

$$\tilde{\phi} := W * \phi, \quad \text{i.e.} \quad \tilde{\phi}(t) = \int_0^{2\pi} W(t-s)\phi(s)\, \mathrm{d}s, \quad \forall t \in \mathbb{R}. \tag{6.1}$$

The Fourier coefficients $\gamma := \mathscr{F}\phi$, $\tilde{\gamma} := \mathscr{F}\tilde{\phi}$, and $\omega := \mathscr{F}W$ of the given, the smoothed, and the window function, respectively, are then related by multiplication

$$\tilde{\gamma}_k = \omega_k \gamma_k, \quad \forall k \in \mathbb{Z}. \tag{6.2}$$

A great number of window functions have been proposed in the mathematical and in the engineering literature [5,22,24,26,32]. Although most of these smoothing processes have originally been designed for the modification of partial sums of slowly converging Fourier series, they are in practice often applied to trigonometric interpolants too.

A very different approach has led to smoothing by spline functions [3,9,11,29,30,31,34,35], which in the case of periodic data on an equispaced mesh can be treated with nearly the same formalism, however. In fact it turns out to be a special case of *smoothing the data* $x \in \Pi_N^{\mathrm{R}}$ by convolution with a *weight* or *window sequence* $w \in \Pi_N^{\mathrm{R}} \cap \Pi_N^{\mathrm{S}}$:

$$\tilde{x} := w * x, \quad \text{i.e.} \quad x_k = \sum_{j=1}^{N} w_{k-j} x_j, \quad \forall k \in \mathbb{Z}. \tag{6.3}$$

The DFT coefficients $c := \mathscr{F}_N x$, $\tilde{c} := \mathscr{F}_N \tilde{x}$, and $\omega := \mathscr{F}_N w$ of the given data, the smoothed data and the weight sequence, respectively, are related by multiplication too [8]:

$$\tilde{c}_k = \omega_k c_k, \quad \forall k \in \mathbb{Z}. \tag{6.4}$$

In contrast to (6.2) the sequence ω is periodic here.

If we either apply the smoothing process (6.1) to an image $\phi := Px$ of P or define $\tilde{\phi} := P(w * x)$ as the image of smoothed data, $\mathrm{D}^m K\phi$ has the Fourier coefficients

$$\tilde{\delta}_k := \tilde{\delta}_k^{(m)} := \mathrm{i}^{m-1} k^m \operatorname{sign}(k)\, \omega_k \tau_k c_k, \quad \forall k \in \mathbb{Z}. \tag{6.5}$$

By substituting $\tilde{\tau}_k := \omega_k \tau_k$ we see that in both cases smoothing is equivalent to using a (noninterpolatory) operator $\tilde{P}$ instead of P. This implies that we can evaluate $\mathrm{D}^m K\tilde{\phi}$ on the grid θ' again with the algorithm of Section 3.

Example 6.1. Cesaro and Lanczos smoothing. The lth Cesaro or Fejer sum uses $\omega_k := 1 - |k|/l$ if $|k| < l$. The Lanczos factors [24] are defined by $\omega_k := \sin(k\pi/l)/(k\pi/l)$ if $|k| < l$. In both cases $\omega_k := 0$ for $|k| \geqslant l$. For the application to the trigonometric interpolant one chooses $l := n$ or $n + 1$. The corresponding window functions as well as comparisons with other windows are given in [32].

Example 6.2. Smoothing by spline functions was proposed by Schoenberg [34], generalized by Anselone and Laurent [3], and further developed mainly by Reinsch [11,29,30,31] and Wahba [9,35]. Given the space

$$H^r := \{ X \in C^{r-1} : \mathrm{D}^{r-1}X \text{ abs. cont.}, \mathrm{D}^r X \in L_2 \} \tag{6.6}$$

with the semi-norm

$$J:\ X \mapsto J(X) := \|\mathrm{D}^r X\|_2 \tag{6.7}$$

($\|\cdot\|_2$ is the L_2-norm), the periodic smoothing spline $S_{N,2r;\rho}$ of degree $2r - 1$ may be defined as the solution of the minimization problem

$$\text{minimize } \{ J^2(X) + \rho R(X) : X \in H^r \}, \tag{6.8}$$

where $\rho > 0$ is a (fixed) parameter and

$$R(X) := \frac{1}{N} \sum_{k=1}^{N} [X(\theta_k) - x_k]^2 \tag{6.9}$$

is the mean-square interpolation error. It is well known [3,34] that $S_{N,2r;\rho}$ is actually a spline function of degree $2r - 1$ satisfying (5.1a) and (5.1c). Moreover, $S_{N,2r;\rho}$ is known to be also the solution of the restricted minimization problem

$$\text{minimize } \{ J^2(X) : X \in H^r,\ R(X) \leqslant \sigma \}. \tag{6.10}$$

If $\bar{\sigma}$ denotes the minimum value that R attains on the set of polynomials of degree $r - 1$, then there belongs to every $\sigma \in (0, \bar{\sigma})$ a positive value of ρ such that $S_{N,2r;\rho}$ is the solution of (6.10). The relationship between σ and ρ is explicitly known in the form $\sigma = [F(\rho)]^2$, where F is a convex strictly decreasing function. Newton's iteration applied to $1/F(\rho) - \sigma^{-1/2} = 0$ allows one to solve it for ρ very efficiently ($1/F$ is concave) [30]. An appropriate choice of ρ on the basis of statistical evidence was proposed by Wahba [35] and Craven and Wahba [9].

To compute the smoothing spline and its conjugate function we still have to determine the coefficients ω_k needed in (6.4) and (6.5). The attenuation factor theory yields the following short derivation. (With more effort the same formula can be derived from the general theory [11,30], cf. [9].) First, we get for every spline function $\tilde{\phi} \in P(\Pi_N^{\mathrm{R}})$

$$J^2(\tilde{\phi}) = \| D^r\tilde{\phi} \|_2^2 = \sum_{k \neq 0} k^{2r} \tau_k^2 |\tilde{c}_k|^2.$$

Since $\tilde{c} \in \Pi_N^{\mathrm{S}}$, relations (2.5), (5.5), (5.2), and (6.4) yield

$$J^2(\tilde{\phi}) = \sum_{k=1}^{N-1} k^{2r} \tau_k \omega_k^2 |c_k|^2. \tag{6.11}$$

Second, due to Parseval's theorem for the DFT [8] and (6.4),

$$R(\tilde{\phi}) = \sum_{k=0}^{N-1} |c_k - \tilde{c}_k|^2 = \sum_{k=0}^{N-1} [1-\omega_k]^2 |c_k|^2. \tag{6.12}$$

Hence,

$$J^2(\tilde{\phi}) + \rho R(\tilde{\phi}) = \sum_{k=0}^{N-1} \left[k^{2r}\tau_k\omega_k^2 + \rho(1-\omega_k)^2\right] |c_k|^2, \tag{6.13}$$

and this quadratic form attains its minimum iff

$$\omega_k = \rho/\left(\rho + k^{2r}\tau_k\right), \quad k = 0, \ldots, N-1. \tag{6.14}$$

Summarizing we get

Theorem 6.1. *To compute for some $m \geqslant 0$ the mth derivative $\mathrm{D}^m KS_{N,2r;\rho}$ of the conjugate function of the smoothing spline $S_{N,2r;\rho}$ on the mesh θ' one applies the algorithm of Theorem* 3.1 *except that $c := \mathscr{F}_N x$ must be modified according to* (6.4), *where $\omega \in \Pi_N^{\mathrm{R}}$ is given by* (6.14). *The factors τ_k and $\kappa_k^{(m)}$ are the same as in Example* 5.2.

7. Optimality of the conjugate function of a periodic spline

In this section we assume that the periodic sequence $x \in \Pi_N^{\mathrm{R}}$ consists of values $x_k = X(\theta_k)$ of a given function X, which for some $r \geqslant 1$ belongs to the Sobolev space H^r defined by (6.6) and the norm $X \mapsto |X| := (X, X)^{1/2}$ based on the inner product

$$(X, Y) := \sum_{j=0}^{r} \int_0^{2\pi} (\mathrm{D}^j X)(t)(\mathrm{D}^j Y)(t)\, \mathrm{d}t = \sum_{j=0}^{r} (\mathrm{D}^j X, \mathrm{D}^j Y)_2. \tag{7.1}$$

[Here $(\cdot,\cdot)_2$ denotes the inner product in L_2. Another possible inner product in H^r is $(X, Y) := (\mathrm{D}^r X, \mathrm{D}^r Y)_2 + \int X(s)\, \mathrm{d}s \cdot \int Y(t)\, \mathrm{d}t$.] We make also use of the semi-norm J defined by (6.7). Our aim is to point out that several optimality criterions for spline interpolants extend naturally to their conjugate functions. The result could be extended to splines with unequal deficiencies, cf. [31]. We start with the *optimality in the sense of Sard and Schoenberg* [33] of the spline $S_{N,2r,q}$ (cf. Ex. 5.5); the case $q = 1$ corresponds to $S_{N,2r}$ of Example 5.2.

Theorem 7.1. *Among all approximations to the functional $X \in H^r \mapsto (\mathrm{D}^m KX)(t)$ (with fixed $0 \leqslant m \leqslant r-1$, $t \in \mathbb{R}$) that are of the form*

$$L_t(X) := \sum_{k=1}^{N} \sum_{j=1}^{q} l_{kj}(t)(\mathrm{D}^{j-1}X)(\theta_k) \tag{7.2}$$

(*where $1 \leqslant q \leqslant r$*) *the one based on replacing X by $S_{N,2r,q}$ is optimal in the following sense*:

$$\sup\{\, |(\mathrm{D}^m KX)(t) - L_t(X)|/J(X) : X \in H^r,\ J(X) \neq 0\}$$

is minimal iff $L_t(X) = (\mathrm{D}^m KS_{N,2r,q})(t)$.

Proof. We only verify that the assumptions of the abstract approach to spline functions as introduced by Attéia, Anselone, and Laurent (see [3] and references given there) and those of Reinsch's elegant proof of the corresponding optimality criterion [31] hold. In fact, H^r is a Hilbert space (namely, a subspace of codimension r of the corresponding Sobolev space of nonperiodic functions defined on $[0, 2\pi]$), and the functionals

$$X \mapsto (\mathrm{D}^j X)(\theta_k), \qquad j=0,\ldots,q-1, \quad k=1,\ldots,N, \tag{7.3}$$

and

$$X \mapsto (\mathrm{D}^m KX)(t), \qquad 0 \leqslant m \leqslant r-1, \quad t \in \mathbb{R}, \tag{7.4}$$

are bounded: If $\xi = \mathscr{F}X$, then $|\mathrm{D}^j X(\theta_k)|$ and $|(\mathrm{D}^j KX)(t)|$ are both bounded by

$$\begin{aligned} |\xi_0| + \sum_{k\neq 0} |k^j \xi_k| &\leqslant |\xi_0| + \left[\sum_{k\neq 0} k^{2r}|\xi_k|^2\right]^{1/2} \left[\sum_{k\neq 0} k^{2j-2r}\right]^{1/2} \\ &\leqslant \|X\|_2 + \|\mathrm{D}^r X\|_2 [2\zeta(2j-2r)]^{1/2} \\ &\leqslant \|X\| [2\zeta(2j-2r)]^{1/2}, \end{aligned} \tag{7.5}$$

where ζ denotes Riemann's zeta function. Moreover, the kernel of the mapping $\mathrm{D}^r\colon H^r \to L_2$ (consisting of all constant functions) and the kernels of the functionals (7.3) have only the zero function in common. Consequently, relation (12) in [31] implies Theorem 7.1.

Reinsch [31] also pointed out that the *optimality in the sense of Golomb and Weinberger* [16] and the related optimality criterion for smoothing splines hold still for abstract splines. In the first case only those $X \in H^r$ with preassigned

$$(\mathrm{D}^j X)(\theta_k) = x_k^{(j)}, \qquad j=0,\ldots,q, \quad k=1,\ldots,N \tag{7.6}$$

are considered, and we have to assume that no constant function X satisfies (7.6). Then, the general theory yields the following theorem.

Theorem 7.2. *The approximation of* $\mathrm{D}^m KX$ *by* $\mathrm{D}^m KS_{N,2r,q}$ *is also optimal in the sense that for fixed* t *and* m $(0 \leqslant m \leqslant r-1)$

$$\inf_{\Lambda_{t,m}\in\mathbb{R}} \sup\{\, |(\mathrm{D}^m KX)(t) - \Lambda_{t,m}|/J(X) : X \in H^r, \text{(7.6) holds}\}$$

is attained for (and only for) $\Lambda_{t,m} = (\mathrm{D}^m KS_{N,2r,q})(t)$.

In the second case we define in generalization of (6.9)

$$R(X) := \sum_{k=1}^{N} \sum_{j=0}^{q-1} \pi_{k,j} \left[\mathrm{D}^j X(\theta_k) - X_k^{(j)}\right]^2,$$

where the $\pi_{k,j}$ are fixed positive weights. The smoothing spline $S_{N,2r,q;\rho}$, which is again defined as the unique solution of the minimization problem (6.8), then satisfies the following optimality criterion [31].

Theorem 7.3. *If* t *and* m $(0 \leqslant m \leqslant r-1)$ *are fixed,*

$$\inf_{\Lambda_{t,m}\in\mathbb{R}} \sup_{X\in H^r} |(\mathrm{D}^m KX)(t) - \Lambda_{t,m}|/\left[J^2(X) + \rho R(X)\right]^{1/2}$$

is attained for (and only for) $\lambda_{t,m} = (\mathrm{D}^m KS_{N,2r,q,\rho})(t)$.

Finally, we point out three additional properties following immediately from $J(X) = J(KX)$ and from three well-known theorems on spline functions, namely the first integral relation, the minimum norm property, and the best approximation property [2, pp. 155–159; 10; 33].

Theorem 7.4. *For $S := S_{N,2r,q}$ and for every $X \in H^r$ fulfilling the interpolation conditions (7.6) the relations*

$$J^2(KX) = J^2(KS) + J^2(KX - KS), \qquad J(KS) \leqslant J(KX),$$

hold, and, if $\hat{S}$ is a spline function satisfying the first and the last condition in (5.19), one has

$$J(KX - KS) \leqslant J(KX - K\hat{S}).$$

8. Functions with jump singularities at mesh points; a basis for the space of functions conjugate to splines

We assume that X is a 2π-periodic function having an absolutely continuous derivative of order $r-1$ in each subinterval (θ_k, θ_{k+1}) and jumps

$$\Delta_k^{(m)} := \mathrm{D}^m X(\theta_k + 0) - \mathrm{D}^m X(\theta_k - 0), \quad m = 0, \ldots, r-1,$$

at the mesh points θ_k, $-\infty < k < \infty$; moreover, we suppose that $\mathrm{D}^r X \in L_2$. To take these jumps into account we apply a well-known technique [12,15]: Consider the 2π-periodic real functions

$$H_m(t) := -\frac{1}{2\pi} \sum_{k \neq 0} \frac{\mathrm{e}^{ikt}}{(ik)^m}, \qquad t \in \mathbb{R}, \quad m \in \mathbb{Z}^+, \tag{8.1a}$$

where in case $m = 1$ the sum has to be understood as a principal value. In real notation,

$$H_{2l-1}(t) = \frac{(-1)^l}{\pi} \sum_{k=1}^{\infty} \frac{\sin kt}{k^{2l-1}},$$

$$H_{2l}(t) = \frac{(-1)^{l+1}}{\pi} \sum_{k=1}^{\infty} \frac{\cos kt}{k^{2l}}, \quad l \in \mathbb{Z}^+. \tag{8.1b}$$

These functions are related to the Bernoulli polynomials B_m [1, p. 804]:

$$H_m(t) = \frac{(2\pi)^{m-1}}{m!} B_m\left(\frac{t}{2\pi}\right) \quad \text{for} \quad t \in (0, 2\pi). \tag{8.2}$$

$H_1(t) = (t - \pi)/(2\pi)$ if $0 < t < 2\pi$. Obviously, $\mathrm{D}H_m = H_{m-1}$ for $m > 2$, and $\mathrm{D}^{m-1} H_m(t) = H_1(t)$ if $t/2\pi \notin \mathbb{Z}$. Since H_1 has jumps of height -1 at the points $2\pi k$ and $H_1''(t) = 0$ elsewhere, we conclude that

$$H(t) := X(t) + \sum_{k=0}^{N-1} \sum_{m=0}^{r-1} \Delta_k^{(m)} H_{m+1}(t - \theta_k) \tag{8.3}$$

lies in the space H^r defined in by (6.6) and (7.1). Hence, the results of Section 7 suggest to approximate H by the spline function $S := S_{N,2r}$ (or by the deficient spline function $S := S_{N,2r,r-1}$ if $\mathrm{D}^m X(\theta_k + 0)$ is given in addition to $\Delta_k^{(m)}$, $1 \leqslant m \leqslant r-1$). Then, $KS_{N,2r}$ can be evaluated on the grid θ' as described in Theorem 1 and Example 5.2 (or Example 5.5, respectively). There only remains the problem to compute the conjugate function of the sum in (8.3).

When transformed into the space of Fourier coefficients, K and the shift operator E are both infinite diagonal matrices, cf. (1.1) and (8.11) below. Hence, they commute, and therefore, the conjugate function of $E^{-k}H_m$: $t \mapsto H_m(t-\theta_k)$ is $E^{-k}G_m$: $t \mapsto G_m(t-\theta_k)$, where

$$G_m := KH_m, \quad m \geqslant 1. \tag{8.4}$$

In view of (8.1)

$$G_{2l-1}(t) \sim \frac{(-1)^{l+1}}{\pi} \sum_{k=1}^{\infty} \frac{\cos kt}{k^{2l-1}},$$

$$G_{2l}(t) \sim \frac{(-1)^{l+1}}{\pi} \sum_{k=1}^{\infty} \frac{\sin kt}{k^{2l}}, \quad l \in \mathbb{Z}^+. \tag{8.5}$$

In particular, G_m is even (odd) if m is odd (even), $\mathrm{D}G_m = G_{m-1}$ if $m \geqslant 2$, and

$$\mathrm{D}^{m-1}G_m(t) = G_1(t) = \frac{-1}{\pi} \log |2 \sin \tfrac{1}{2}t| \quad \text{if} \quad \tfrac{1}{2}t/\pi \notin \mathbb{Z}. \tag{8.6}$$

Therefore, these functions G_m could be evaluated by integration if $m \geqslant 2$:

$$G_{2l}(t) = \int_0^t G_{2l-1}(s)\, \mathrm{d}s,$$

$$G_{2l+1}(t) = \int_0^t G_{2l}(s)\, \mathrm{d}s + \frac{(-1)^l}{\pi}\zeta(2l+1), \quad l \in \mathbb{Z}^+. \tag{8.7}$$

Finally, since H has been approximated by S

$$KX(t) \approx KS(t) - \sum_{k=0}^{N-1} \sum_{m=0}^{r-1} \Delta_k^{(m)} G_{m+1}(t-\theta_k). \tag{8.8}$$

For the error of this approximation, the bounds given in Section 9 below will hold with X replaced by H.

Another way to handle the given problem is to modify the DFT coefficients $c = \mathscr{F}_N X(\theta)$ instead of modifying X. In fact, since the aliasing formula (2.7) holds whenever the Fourier series converges at each θ_k, the discrete Fourier coefficients of H_m $(m \geqslant 1)$ are just

$$(\mathscr{F}_N H_m(\theta))_0 = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ -\zeta(m)/\pi(\mathrm{i}N)^m & \text{if } m \text{ is even,} \end{cases} \tag{8.9}$$

$$(\mathscr{F}_N H_m(\theta))_k = \frac{-1}{2\pi(\mathrm{i}k)^m} \sigma_{m-1}(k/N), \quad k = 1, \ldots, N\text{-}1.$$

(For $m = 1$ these coefficients deviate from Ehlich's [12], who defines $H_1(0) := -\tfrac{1}{2}$.) By (8.5) we obtain similarly

$$(\mathscr{F}_{\nu N} G_m(\theta'))_0 = \begin{cases} 0 & \text{if } m \text{ is even,} \\ \mathrm{i}\zeta(m)/\pi(\mathrm{i}\nu N)^m & \text{if } m \text{ is odd,} \end{cases}$$

$$(\mathscr{F}_{\nu N} G_m(\theta'))_k = \frac{\mathrm{i}}{2\pi(\mathrm{i}k)^m}\left[1 + \psi_{m-1}\left(\frac{k}{\nu N}\right) - \psi_{m-1}\left(\frac{-k}{\nu N}\right)\right], \quad k = 1, \ldots, \nu N - 1. \tag{8.10}$$

Moreover, as is well known, for E defined by (2.2b)

$$\left[\mathscr{F}_N\left(E^{-l}H_m\right)(\theta)\right]_k = \mathrm{e}^{-2\pi i lk/N}\left[\mathscr{F}_N H_m(\theta)\right]_k,$$
$$\left[\mathscr{F}_{\nu N}\left(E^{-l}G_m\right)(\theta')\right]_k = \mathrm{e}^{-2\pi i lk/N}\left[\mathscr{F}_{\nu N} G_m(\theta')\right]_k. \tag{8.11}$$

[Note that $(E^{-l}G_m)(\theta') = E^{-\nu l}(G_m(\theta'))$ if at right E is defined by (2.2a).]

Formulas (8.9)–(8.11) allow one to compute the discrete Fourier transforms $\mathscr{F}_N$ and $\mathscr{F}_{\nu N}$ of the sums in (8.3) and (8.8), respectively. So, the singularities could also be taken into account by modifying $\mathscr{F}_N X(\theta)$.

Finally, we recall the well-known fact that the functions

$$\phi_0(t) :\equiv 1,$$
$$\phi_k(t) :\equiv (E^{-k}-1)H_{2r}(t) = H_{2r}(t-\theta_k) - H_{2r}(t), \quad k = 1,\ldots,N-1,$$

are a basis of the space $\mathscr{S}_{N,2r}$ of periodic spline functions $S_{N,2r}$ of order $2r$ with knots on θ. Equation (1.1) shows that the kernel of K consists of all constant functions. Consequently, *the functions*

$$K\phi_k = (E^{-k}-1)G_{2r}, \quad k = 1,\ldots,N-1, \tag{8.12}$$

are a basis of the conjugate space $K\mathscr{S}_{N,2r}$.

9. Error bounds and asymptotic convergence

Error bounds for spline interpolation are a well covered subject, see [2,12,14,15,28,36]. Some of these results also apply to the conjugate function of a spline, in particular, those that are based on a term by term estimate of the Fourier series of the error $X - Px$. We do not attempt to cover this subject completely, but we present some examples of such estimates.

For $X \in L_2$ let $\xi = \mathscr{F}X \in l_2$ be the Fourier coefficients of X. We generalize the definition (6.6) of H^s to arbitrary real $s > \frac{1}{2}$:

$$H^s := \left\{X \in \Phi : \left(|k|^s |\xi_k|\right)_{k=-\infty}^{\infty} \in l_2\right\}, \tag{9.1a}$$

but we use again only a semi-norm:

$$\|X\|_{H^s} := \left(\sum_{k=-\infty}^{\infty} |k|^{2s} |\xi_k|^2\right)^{1/2}. \tag{9.1b}$$

Similarly, Φ^s is now defined for arbitrary $s \geqslant 0$:

$$\Phi^s := \left\{X \in \Phi : \left(|k|^s |\xi_k|\right)_{k=-\infty}^{\infty} \in l_1\right\}, \tag{9.2a}$$

$$\|X\|_{\Phi^s} := \sum_{k=-\infty}^{\infty} |k|^s |\xi_k|. \tag{9.2b}$$

(Here, let $|k|^0 := 1$, $\forall k \in \mathbb{Z}$.) Trivially, $\Phi^0 = \Phi$, $\Phi^s \subset \Phi^{s'}$ if $0 \leqslant s' \leqslant s$, and $\Phi^s \subset H^s \subset H^{s''}$ if $\frac{1}{2} < s'' \leqslant s$. For every integer $m \in [0, s]$, $X \in H^s$ implies $\mathrm{D}^m X \in L_2$, and $X \in \Phi^s$ implies $\mathrm{D}^m X = \tilde{X} \in \Phi$ a.e.; so, $\|\mathrm{D}^m X\|_\infty$ is well defined for $X \in \Phi^s$, and

$$\|\mathrm{D}^m X\|_\infty \leqslant \|X\|_{\Phi^m} \leqslant \|X\|_{\Phi^s}, \quad 0 \leqslant m \leqslant s. \tag{9.3}$$

If $X \in H^s$ $(s > \frac{1}{2})$ and $0 < \epsilon \leqslant s - \frac{1}{2}$, then

$$\sum_{k \neq 0} |k|^{s-\epsilon-1/2} |\xi_k| \leqslant \Big(\sum_{k \neq 0} |k|^{2s} |\xi_k|^2 \Big)^{1/2} \Big(\sum_{k \neq 0} |k|^{-1-2\epsilon} \Big)^{1/2} < \infty. \tag{9.4}$$

Consequently, $\Phi^s \subset H^s \subset \Phi^{s-\epsilon-1/2}$ [14, p. 48] and, if $\delta_{s,r}$ is Kronecker's symbol,

$$\| X \|_{\Phi^{s-\epsilon-1/2}} - \delta_{s,\epsilon+1/2} |\xi_0| \leqslant \| X \|_{H^s} [2\zeta(1+2\epsilon)]^{1/2}. \tag{9.5}$$

We also conclude that in the definition of H^s the restriction $X \in \Phi$ is not essential: every $X \in L_2$ with $(|k|^s |\xi_k|) \in l_2$ for some $s > \frac{1}{2}$ satisfies $(|\xi_k|) \in l_1$ and is therefore a.e. equal to an element of H^s.

The following simple lemma, which is an immediate consequence of (1.1), motivates our interest in the spaces H^s and Φ^s.

Lemma 9.1. *$K(L_2) \subset L_2$, $K(H^s) \subset H^s$ $(s > \frac{1}{2})$, and $K(\Phi^r) \subset \Phi^r$ $(r \geqslant 0)$. The corresponding operator norms of K equal 1: $\| K \|_2 = \| K \|_{H^s} = \| K \|_{\Phi^r} = 1$. Moreover, $\| KX \|_{H^s} = \| X \|_{H^s}$ $(\forall X \in H^s)$ and $\| KX \|_{\Phi^r} = \| X \|_{\Phi^r}$ $(\forall X \in \Phi^r)$ if $r > 0$.*

(Note that since on the orthogonal complement of the kernel of K our semi-norms are actually norms, $\| K \|_{H^s}$ and $\| K \|_{\Phi^r}$ are both well-defined.)

Lemma 9.1 implies that in any of the three norms any known error bound for $X - Px$ is also an error bound for $KX - KPx$. In particular, this applies to *mean convergence*, i.e. to the L_2-norm of the error. For X, $\phi \in H^s$ and $0 \leqslant m \leqslant s$ one has clearly

$$\| \mathrm{D}^m KX - \mathrm{D}^m K\phi \|_2 \leqslant \| \mathrm{D}^m X - \mathrm{D}^m \phi \|_2 \leqslant \| X - \phi \|_{H^s}. \tag{9.6}$$

If $m = s$ (> 0), equality holds. Error bounds and asymptotic estimates for $\| \mathrm{D}^m X - \mathrm{D}^m S_{N,2r} \|_2$ were established by Golomb [14, Section 8]. For example, if $r \geqslant 1$, $0 \leqslant m \leqslant 2r - 1$, $m + \frac{1}{2} < s \leqslant 2r$ and $X \in H^s$, then [14, (8.14)]

$$\begin{aligned} \| \mathrm{D}^m X - \mathrm{D}^m S_{N,2r} \|_2 &\leqslant 2^{s+2} \big[1 + 3(2s - 2m - 1)^{-1/2} \big] N^{m-s} \| X \|_{H^s} \\ &= \mathrm{O}(N^{m-s}) \quad \text{as } N \to \infty, \end{aligned} \tag{9.7}$$

while for $X \in H^r$ [14, Theorem 8.2]

$$\| \mathrm{D}^r X - \mathrm{D}^r S_{N,2r} \|_2^2 = \| \mathrm{D}^r X \|_2^2 - \| \mathrm{D}^r S_{N,2r} \|_2^2 = \mathrm{o}(1) \quad \text{as } N \to \infty. \tag{9.8}$$

Assuming $X \in C^r$ or even $X \in C^{2r}$ Werner and Schabak [36, pp. 166–169] gave simple but improved bounds for $\| X - S_{N,2r,q} \|_2$, $1 \leqslant q \leqslant r$, which could be generalized to derivatives of the error.

To estimate the rate of *uniform convergence* we assume X, $\phi \in \Phi^s$, $s - \frac{1}{2} > m \geqslant 0$ and apply Lemma 9.1 and the inequalities (9.3) and (9.5) to $KX - K\phi$. In (9.5) ξ_0 is replaced by $-\mathrm{i}\ \mathrm{sign}\ (0)[\xi_0 - \gamma_0] = 0$. Hence, we get

Lemma 9.2. *Assume X, $\phi \in \Phi^s$, $0 \leqslant m < s - \frac{1}{2}$, and let $\epsilon := s - m - \frac{1}{2}$. Then*

$$\| \mathrm{D}^m KX - \mathrm{D}^m K\phi \|_\infty \leqslant \| KX - K\phi \|_{\Phi^m} \leqslant \| X - \phi \|_{H^s} [2\zeta(1+2\epsilon)]^{1/2}.$$

For example, if we suppose $m + \frac{3}{2} < s \leqslant 2r$ and apply Lemma 9.2 with s replaced by $m + 1$, we may use estimate (9.7) (with m replaced by $m+1$) to obtain

$$\begin{aligned}\|\mathrm{D}^m KX - \mathrm{D}^m KS_{N,2r}\|_\infty &\leqslant \|X - S_{N,2r}\|_{H^{m+1}} 3^{-1/2}\pi \\ &\leqslant 2^{s+2}3^{-1/2}\pi\left[1 + 3(2s-2m-3)^{-1/2}\right] N^{m-s+1}\|X\|_{H^s} \\ &= \mathrm{O}(N^{m-s+1}).\end{aligned} \tag{9.9}$$

However, both for $X \in \Phi^s$ and $X \in H^s$ this error bound is not of the optimal order. Better bounds are obtained by adapting Golomb's estimates of the maximum error of spline interpolants [14, Sections 6 and 7] to the conjugate function. Instead of presenting these calculations in detail we only state the basic ideas and some of the intermediate results.

Let $b_\mu := S_{N,2r}$ be the spline function interpolating $X(t) := \mathrm{e}^{\mathrm{i}\mu t}$. Since $c_k = \delta_{\mu,l}$, $k \equiv l \pmod N$, (2.3) and (5.5) yield

$$\begin{aligned} b_\mu(t) &= \sum_{k=-\infty}^{\infty} \tau_{\mu+kN}\, \mathrm{e}^{\mathrm{i}(\mu+kN)t} \\ &= \tau_\mu \sum_{k=-\infty}^{\infty} (1 + kN/\mu)^{-2r}\, \mathrm{e}^{\mathrm{i}(\mu+kN)t}, \quad \mu \not\equiv 0 \pmod N,\end{aligned} \tag{9.10}$$

where $\tau_\mu = [\sigma_{2r-1}(\mu/N)]^{-1}$. Hence, for $m = 0, \ldots, 2r-2$,

$$\mathrm{D}^m Kb_\mu(t) = -\mathrm{i}(\mathrm{i}\mu)^m \tau_\mu \sum_{k=-\infty}^{\infty} \operatorname{sign}(\mu + kN)(1 + kN/\mu)^{m-2r}\mathrm{e}^{\mathrm{i}(\mu+kN)t}. \tag{9.11}$$

Obviously, $b_{\mu+kN} = b_\mu = -\overline{b}_\mu$ and $b_0(t) \equiv 1$. So, we can concentrate on $0 \leqslant \mu < N$. A term by term estimate in (9.11) leads to the following analogue of Lemmata 6.1 and 6.2 in [14]. (For the proof of $\beta_\mu^{(m)} < 3$ see [14, pp. 37–39].)

Lemma 9.3. *If* $0 \leqslant \mu < N$ *and* $0 \leqslant m \leqslant 2r-2$,

$$\|\mathrm{D}^m Kb_\mu\|_\infty \leqslant \mu^m \beta_\mu^{(m)} < 3\mu^m,$$

where $\beta_0^{(m)} := 0$ *and*

$$\beta_\mu^{(m)} := \sum_{k=-\infty}^{\infty} |1 + kN/\mu|^{m-2r} \Big/ \sum_{k=-\infty}^{\infty} [1 + kN/\mu]^{-2r}, \quad 0 < \mu < N.$$

In particular, $\|Kb_\mu\|_\infty \leqslant 1$.

Remark. Note that $\mathrm{D}^{2r-1}Kb_\mu$ is unbounded at $t = 2\pi/N$ if $\mu \not\equiv 0 \pmod N$, as can be seen from (9.11). Therefore, in Lemma 9.3 and in the following results m has to be limited to $0 \leqslant m \leqslant 2r-2$, while in [14] the case $m = 2r-1$ is included.

Lemma 9.4. *If* $0 \leqslant \mu < N$, $0 \leqslant m \leqslant 2r-2$, *and* $X(t) = \mathrm{e}^{\mathrm{i}\mu t}$,

$$\|\mathrm{D}^m KX - \mathrm{D}^m Kb_\mu\|_\infty \leqslant \mu^m \alpha_\mu^{(m)} \leqslant 2^{2r+2}\mu^{2r}\tau_\mu N^{m-2r},$$

where $\alpha_0^{(m)} := 0$ *and*

$$\alpha_\mu^{(m)} := \sum_{k \neq 0} \left\{ |1 + kN/\mu|^{m-2r} + [1 + kN/\mu]^{-2r} \right\} \Big/ \sum_{k=-\infty}^{\infty} [1 + kN/\mu]^{-2r}, \quad 0 < \mu < N.$$

Proof. The coefficient of $e^{i\mu t}$ in the Fourier series of $D^m KX - D^m Kb_\mu$ is according to (9.11), (2.8), and (5.5)

$$
\begin{aligned}
-i(i\mu)^m(1-\tau_\mu) &= -i(i\mu)^m \sum_{k\neq 0} \tau_{\mu+kN} \\
&= -i(i\mu)^m \tau_\mu \sum_{k\neq 0} (1+kN/\mu)^{-2r}.
\end{aligned}
$$

Hence, (9.11) yields the asserted uniform bound $\mu^m \alpha_\mu^{(m)}$. For the explicit estimate of $\alpha_\mu^{(m)}$ leading to the inequality on the right-hand side, see [14, pp. 39–41]. However, our factor $\tau_\mu = 1/\sigma_{2r-1}(\mu/N) = 1/\Sigma(1+kN/\mu)^{-2r}$ is an improvement of Golomb's bound in his Lemma 6.3. □

Now, we may proceed as in Section 7 of [14]. For arbitrary $X = \mathscr{F}^{-1}\xi \in \Phi$

$$
S_{N,2r} = \sum_{k=-\infty}^{\infty} \xi_k b_k = \sum_{k=0}^{N-1} c_k b_k, \tag{9.12}
$$

where $c_k = \Sigma \xi_{k+jN}$ ($k \in \mathbb{Z}$) are the DFT coefficients as before. Applying this to a trigonometric polynomial $\tilde{X}$ of degree $M \leqslant N-1$ we get by Lemma 9.4 for $m = 0, \ldots, 2r-2$

$$
\| D^m K\tilde{X} - D^m K\tilde{S}_{N,2r} \|_\infty \leqslant 2^{2r+2} N^{m-2r} \sum_{k=-M}^{M} \tau_k k^{2r} |\xi_k|, \tag{9.13}
$$

cf. Lemma 7.1 in [14]. On the other hand, for $X \in \Phi^s$ ($s \geqslant 0$) Lemma 9.3 and (9.12) imply

$$
\| S_{N,2r} \|_\infty \leqslant \sum_{k=-\infty}^{\infty} |\xi_k| = \| X \|_{\Phi^0}, \tag{9.14a}
$$

$$
\| D^m S_{N,2r} \|_\infty \leqslant 3 \sum_{k=-\infty}^{\infty} |k|^m |\xi_k| = 3 \| X \|_{\Phi^m}, \quad 0 \leqslant m \leqslant \min\{s, 2r-2\}, \tag{9.14b}
$$

cf. Lemma 7.2 in [14]. Finally, $\tilde{X}$ may be thought of as the Mth partial sum of the Fourier series of $X \in \Phi^s$. Starting with

$$
\begin{aligned}
\| D^m K(X - S_{N,2r} \|_\infty \leqslant{} & \| D^m K(X - \tilde{X}) \|_\infty \\
& + \| D^m K(\tilde{X} - \tilde{S}_{N,2r}) \|_\infty + \| D^m K(\tilde{S}_{N,2r} - S_{N,2r}) \|_\infty
\end{aligned}
$$

and applying (9.13) to the second and (9.14) to the third term at right we get, after some calculations (cf. [14], p. 46) and by choosing $M := \lfloor N^{1/2} \rfloor$ if $m = s$ and $M := N-1$ if $m < s$, the following theorem.

Theorem 9.5. *Assume* $X = \mathscr{F}^{-1}\xi \in \Phi^s$, $0 \leqslant s \leqslant 2r$, *and* $0 \leqslant m \leqslant \min\{s, 2r-2\}$. *Then,*

$$
\| D^m KX - D^m KS_{N,2r} \|_\infty \leqslant
\begin{cases}
2^{2r+2}\left\{ N^{m/2-r} \sum_{|k|\leqslant M} |k|^m |\xi_k| + \sum_{|k|>M} |k|^m |\xi_k| \right\} \\
\qquad\qquad \text{if } m = s \text{ and } M := \lfloor N^{1/2} \rfloor, \\
2^{2r+2} N^{m-s} \| X \|_{\Phi^s} \quad \text{if } m < s.
\end{cases}
$$

In particular, as $N \to \infty$,

$$
\| D^m KX - D^m KS_{N,2r} \|_\infty =
\begin{cases}
o(1) & \text{if } m = s, \\
O(N^{m-s}) & \text{if } m < s.
\end{cases}
$$

It is also easy to verify that Golomb's estimate of $\|D^m X - D^m S_{N,2r}\|_\infty$ for $X \in H^s$ [14, Section 9] applies to the conjugate function except when $m = 2r - 1$. We only cite the resulting theorem (cf. [14], (9.8)).

Theorem 9.6. *Assume* $X \in H^s$, $\frac{1}{2} < s \leqslant 2r$, *and* $0 \leqslant m < \min\{s - \frac{1}{2}, 2r - 1\}$. *Then,*

$$\|D^m KX - D^m KS_{N,2r}\|_\infty \leqslant 2^{s+2}\left[\frac{2s-1}{2s-2m-1}\right]^{1/2} N^{m-s+1/2} \|X\|_{H^s} = O(N^{m-s+1/2}) \quad \text{as } N \to \infty.$$

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